Problem Set 2

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Hopfield model

In the Hopfield network, we make even a more drastic assumption that the activity of a neuron is a binary variable, $s_i \in [+1, -1]$. In the deterministic version of this model, the state of a cell is set according to the following equation

$$s_i(t + \Delta t) = \mathbf{sgn}(\sum_j w_{ij} s_j), \tag{1}$$

We consider an asynchronous update rule for each neuron (choosing either a random or fixed update order and update s_i according to that order at each clock cycle).

(1) Check my lecture note and proof that any mixture of an odd number of memory patterns, such as

$$\xi^{mix} = \mathbf{sgn}(\xi^{\mu_1} + \xi^{\mu_2} + \xi^{\mu_3}) \tag{2}$$

are fixed point of the Hopfield model.

(2) Numerically simulate the dynamics of Hopfield network with random unbiased memory patterns, using 500 binary neurons. We want to draw some statistical conclusions, so in the tasks that follow, you should run many simulations using different sets of random stored patterns and, for each, start from many different initial conditions (so I recommend writing reasonably efficient routines, e.g. minimize the number of for-loops in MATLAB). For simplicity, you can choose the order of update of the neurons by going through neuron 1 to N all the time.

i) Probability of perfect recall: Store 10 patterns, run simulations to find the probability of perfect recall. That is, start with a pattern, corrupt some fraction of its bits, input the corrupted pattern to the network, and wait until the network settles into a fixed point. Plot the fraction of times you get perfect recall of the original state against the fraction of corrupted bits.

ii) Memory retrieval with errors: Start the dynamics with one of the memory states. Plot the fraction of errors (fraction of incorrectly recalled bits) in the final state as a function of the number of stored patterns P. Change P from 10 to 100.

Associative Memory with Decaying Memory Traces

Consider a network of binary neurons with the dynamics similar to that of the Hopfield Model.

Network dynamics: The value of each neuron is represented by $S_i = \pm 1$. The input function to each neuron is defined by $h_i(t)$, which stands for the weighted sum of the values of the rest of the neurons in the networks, weighted by the connection matrix W:

$$h_i(t) = \sum_{j=1}^{N} W_{ij} S_j(t) \tag{3}$$

At each time step a single neuron is chosen and its state is updated according to the following rule:

$$S_i(t + \Delta t) = \mathbf{sgn}(h_i(t)) \tag{4}$$

Connectivity Matrix: The memories are stored in a temporal sequence by updating the weight matrix W_{ij} dynamically. We start with $W_{ij} = 0$. As time proceeds, we add a new stored pattern ξ^μ to the existing memory while allowing the other memories to decay exponentially:

$$W_{ij}^t = \lambda W_{ij}^t + \frac{1}{N} \xi_i^\mu \xi_j^\mu \tag{5}$$

($W_{ii} = 0$). We let $\lambda = e^{-\tau}, \tau > 0$. (This kind of memory is dubbed a *palimpsest* — papyrus that has been written on more than once, with the earlier writing incompletely erased and often legible.)

- Show that the connection matrix can be written as:

$$W_{ij}^p = \frac{1}{N} \sum_{\mu=0}^{p} e^{-\mu t} \xi_i^\mu \xi_j^\mu, i \neq j \tag{6}$$

- Consider a steady state situation where the time step p approaches infinity. Estimate how many past memories we can store, assuming all ξ_i^μ are drawn independently with equal probability of ± 1.

Hint: To estimate the memory capacity, use a 'signal-to-noise' analysis similar to that presented in the lecture for the Hopfield Model. A pattern ξ^μ is stable if for all neurons $\xi_i^\mu h_i^\mu > 0$.

1. Derive an expression for the mean and variance of this quantity.
2. Assuming that N and p large, approximate the distribution of $\xi_i^p h_i^p$ by a normal distribution. Find the maximum p for which the probability of error (instability of the stored pattern) is small. Note: For the sake of calculation, it is useful to define new variables $\bar{\tau}$ and $\bar{p}$ with $\tau = \frac{\bar{\tau} \ln N}{N}$, $p = \frac{\bar{p} N}{\ln N}$, (so that in particular p is expected to be large and τ is expected to be small) to capture the expected dependence on N.
3. Plot the maximal value of $\bar{p}$ as a function of $\bar{\tau}$ and discuss the results and their implications on the performance of this model in comparison to the Hopfield model.